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Analysis of non-stationary temperature fields in laminated strips and plates

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Abstract

A method for determination of thermal condition of laminated elements of structures is offered. The method is based on a presentation of temperature distribution through the thickness of each layer by means of orthonormal Legendre polynomials. As a numerical example, a solution of the non-stationary heat conduction problem for laminated strips and plates is obtained. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Laminated strips and plates; Non-stationary heat conduction; Temperature distribution

1. Introduction

Changes of temperature are often the cause of failure of structures, especially multilayer ones, when temperature difference between internal and external surfaces is significant.

The majority of publications, devoted to thermoelasticity of laminated structures, deal with deformation of such structures under conditions of steady temperature fields (Wu and Tauchert, 1980; Reddy and Hsu, 1980; Khdeir, 1997; Verijenko et al., 1999) or dynamic temperature fields with prescribed distribution through the thickness (Shuji and Masataka, 1991; Heuer et al., 1992). The hypothesis about a piecewise-linear temperature distribution through the thickness of a laminated package is often applied. However, the non-stationary character of a problem requires a more exact description of the temperature field obtained directly from solution of a heat conduction equation.

Savoia and Reddy (1995) used the quasi-static theory of thermoelasticity to examine stresses in multilayer rectangular simply supported plates affected by thermal and mechanical loads. The authors applied polynomial and exponential temperature distributions through the thickness of each layer which allow to consider steady-state and transient thermal conditions. The temperature distribution over top and bottom surfaces of plates is given.

Reddy and Chin (1998) have numerically analysed thermomechanical behaviour of functionally graded cylinders and plates under transient thermal loading conditions. Temperature and stress fields are

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determined from the non-linear coupled thermoelastic problem. The results obtained are compared to those from the uncoupled formulation. Temperature and displacements are approximated by the Lagrange interpolation functions.

Tanigawa et al. solved the problems of thermal bending of laminated composite beams (Tanigawa et al., 1991a) and rectangular laminated plates (Tanigawa et al., 1991b) under a heat supply on top and bottom surfaces. Temperature fields are obtained from a solution of the transient heat conduction problem. The one-dimensional case in the direction of thickness is considered for a beam. The three-dimensional heat conduction problem is dealt with for a plate. The Laplace transformation over the time and finite cosine transformation are applied. The authors examined the effect of relaxation on thermal deflection and stresses of the non-homogeneous plates.

Santhosh (1992) determined the distribution of temperature and strains through the thickness in layered media subjected to thermal shock in terms of equations of thermoelasticity including inertia effects. The solution is obtained on the basis of an explicit-implicit finite difference procedure. The numerical results are presented for laminate composites fixed on a foundation. The thermal shock is given for a free surface by Heaviside function. It is shown that for a small time interval the influence of dynamic effect of thermal stresses is significant.

Nusier and Newaz (1998) have obtained a temperature field in a three-layer cylinder under transient heating. The temperature distribution through the thickness of the cylinder results from solution of the heat conduction equation. Over the radial coordinate expansion was applied in terms of Bessel functions.

Kantor et al. (2000) offered a method of solution of the one-dimensional non-stationary heat conduction problem in a laminated medium based on introduction of the temperature distribution in each layer by a system of Legendre polynomials. In the present paper this method is generalized for the case of a laminated strip and laminated rectangular plate with an internal heat source.

2. Non-stationary heat conduction in a laminated strip

A laminated strip of length l assembled from arbitrary number of isotropic layers ($i = \overline{1,N}$, Fig. 1) is being considered.

The heat conduction equation for the ith layer has the form

$$v_i \Delta T = \frac{\partial T}{\partial t}, \quad i = 1, 2, \ldots, n, \ \ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \tag{1}$$

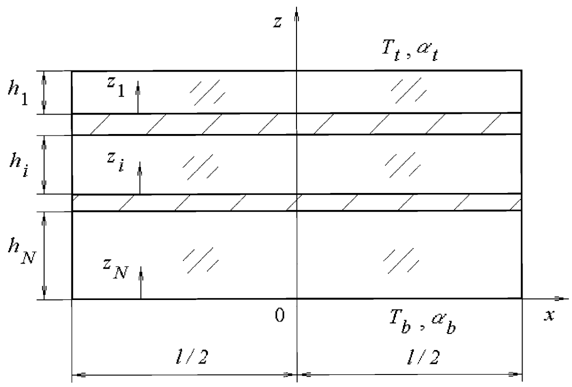

Fig. 1. Multilayer strip.

where $v_i = \lambda_i/(\gamma_i c_i)$ is the thermal diffusivity, λ_i is the thermal conductivity, γ_i is the density of the ith layer material, c_i is the specific heat at constant volume of the ith layer.

For the sake of simplicity, let us consider a symmetrical problem concerning the middle of a strip $x = l/2$, and seek a solution in the form

$$T_i(x, z_i, t) = \sum_{k=1}^{M} \left[a_{ik}(t) f_1(\bar{z}) + b_{ik}(t) f_2(\bar{z}) + c_{ik}(t) f_3(\bar{z}) \right] \cos(\alpha_k x), \tag{2}$$

where $0 \leqslant x \leqslant l/2$, $\alpha_k = 2\pi(k-1)/l$, t is time. Coordinate z_i is measured from an internal surface of each layer.

As functions f_r, $r = 1, 2, 3$, we choose Legendre orthonormal polynomials of the $(r-1)$th degree for the interval $0 \leqslant \bar{z} \leqslant 1$ (Jackson, 1948)

$$f_1 = 1, \quad f_2 = \sqrt{3}\left(2\bar{z} - 1\right), \quad f_3 = \sqrt{5}\left(6\bar{z}^2 - 6\bar{z} + 1\right) \tag{3}$$

$$\int_0^1 f_k f_l \, \mathrm{d}\bar{z} = \delta_{kl}. \tag{4}$$

The following designations are used: $\bar{z} = z_i/h_i$, $0 \leqslant z_i \leqslant h_i$; h_i is the thickness of the ith layer.

Substituting expression (2) in Eq. (1) and projecting the obtained equation to the system of functions f_r (3), we can find the relationship for the kth harmonic of a series (2)

$$v_i \int_0^1 \left\{ \begin{array}{l} a_{ik}\left[f_1''/h_i^2 - \alpha_k^2 f_1\right] f_r \cos(\alpha_k x) \\ b_{ik}\left[f_2''/h_i^2 - \alpha_k^2 f_2\right] f_r \cos(\alpha_k x) \\ c_{ik}\left[f_3''/h_i^2 - \alpha_k^2 f_3\right] f_r \cos(\alpha_k x) \end{array} \right\} \mathrm{d}z = \int_0^1 \left(\dot{a}_{ik} f_1 + \dot{b}_{ik} f_2 + \dot{c}_{ik} f_3 \right) f_r \cos(\alpha_k x) \, \mathrm{d}z,$$

where

$$f_r'' = \frac{\mathrm{d}^2 f_r}{\mathrm{d}z^2}, \quad \dot{a}_{ik} = \frac{\mathrm{d}a_{ik}}{\mathrm{d}t}, \quad \dot{b}_{ik} = \frac{\mathrm{d}b_{ik}}{\mathrm{d}t}, \quad \dot{c}_{ik} = \frac{\mathrm{d}c_{ik}}{\mathrm{d}t}.$$

In view of the orthogonality condition (4) of the system of functions f_r (3), we obtain

$$\dot{a}_{ik} = v_i \frac{12\sqrt{5}}{h_i^2} c_{ik} - v_i \alpha_k^2 a_{ik}, \quad \dot{b}_{ik} = -v_i \alpha_k^2 b_{ik}, \quad \dot{c}_{ik} = -v_i \alpha_k^2 c_{ik}. \tag{5}$$

As initial conditions let us accept

$$T_i(x, z_i, 0) = \text{const}.$$

In the absence of heat exchange, and when the power of heat sources is equal to zero, at $t = 0$, it follows that

$$a_{ik}(0) = T_0, \quad b_{ik}(0) = c_{ik}(0) = 0, \quad i = \overline{1, N}.$$

The boundary conditions of convective heat transfer over the top and bottom surfaces of the strip are

$$\lambda_1 \nabla T_1|_{z_1=h_1} = \alpha_\mathrm{t}\left(T_\mathrm{t} - T_1|_{z_1=h_1}\right), \qquad -\lambda_N \nabla T_N|_{z_N=0} = \alpha_\mathrm{b}\left(T_\mathrm{b} - T_N|_{z_N=0}\right), \tag{6}$$

where α_t and α_b are the convective heat transfer coefficients at the top and bottom of the strip, T_t and T_b are the temperatures of environments over the top and bottom surfaces, respectively.

Taking into account that $\nabla T_i = \frac{\partial}{\partial z} T_i$, $f_1 = 1$, and substituting the relationship (2) into conditions (6), the following equalities are obtained:

$$\frac{\lambda_1}{h_1}\left(b_{1k}2\sqrt{3}+c_{1k}6\sqrt{5}\right)=\alpha_t\left(T_t-a_{1k}-b_{1k}\sqrt{3}-c_{1k}\sqrt{5}\right),$$
$$\frac{\lambda_N}{h_N}\left(b_{Nk}2\sqrt{3}-c_{Nk}6\sqrt{5}\right)=\alpha_b\left(T_b-a_{Nk}+b_{Nk}\sqrt{3}-c_{Nk}\sqrt{5}\right). \tag{7}$$

The conditions of equality of heat flows and temperatures on interfaces of layers are:

$$-\lambda_i\nabla T_i\big|_{z_i=0}+\lambda_{i+1}\nabla T_{i+1}\big|_{z=h_{i+1}}=0,\quad T_i\big|_{z_i=0}-T_{i+1}\big|_{z=h_{i+1}}=0,\quad i=\overline{1,N-1}.$$

It follows from this that

$$-\frac{\lambda_i}{h_i}\left(b_{ik}2\sqrt{3}-c_{ik}6\sqrt{5}\right)+\frac{\lambda_{i+1}}{h_{i+1}}\left(b_{i+1,k}2\sqrt{3}+c_{i+1,k}6\sqrt{5}\right)=0, \tag{8}$$

$$a_{ik}-b_{ik}\sqrt{3}+c_{ik}\sqrt{5}-a_{i+1,k}-b_{i+1,k}\sqrt{3}-c_{i+1,k}\sqrt{5}=0. \tag{9}$$

The joint number of boundary conditions (7) and interface conditions (8) and (9) is equal to $2NM$. The number of unknown time-dependent functions in the given problem is equal to $3NM$. Referring to the second and third Eq. (5), we note that they have zero partial solutions, as the initial temperature field is steady. We reject these equations. It means that the change of functions $b_{ik}(t)$ and $c_{ik}(t)$ with time in approximation (2) is determined by their relationship from $a_{ik}(t)$. Then the number of unknown functions $a_{ik}(t)$, $b_{ik}(t)$ and $c_{ik}(t)$ equals to $3NM$; it is equal to the number of Eq. (5) for $a_{ik}(t)$ and number of conditions (7)–(9), that is $3NM$.

To define the functions $b_{ik}(t)$ and $c_{ik}(t)$, it is necessary to use conditions (7)–(9), expressing them in terms of $a_{ik}(t)$. This method allows the number of equations to be equated to the number of unknowns. The solution obtained on each time step will exactly satisfy the boundary conditions and the conditions of the interface layers. The non-stationary character of the problem will be taken into consideration owing to the functions $a_{ik}(t)$ to be determined from the solution of Cauchy problem.

By virtue of linearity of the problem, the system of equations breaks up into M independent systems. If a heat-generating film with power $W(x)H(t)$ is placed between the ith and $(i+1)$th layers, it is necessary to take into account, on the right-hand side of the appropriate condition (8), a harmonic

$$W_k=\frac{2}{l}\int_0^{l^*/2}W(x)\cos(\alpha_k x)\,\mathrm{d}x$$

multiplied by $H(t)$, where $H(t)$ is Heaviside function.

In the specific case when

$$W(x)=W\quad\text{at}\quad 0\leqslant|x|\leqslant l^*/2,$$
$$W(x)=0\quad\text{at}\quad l^*/2\leqslant|x|\leqslant l/2, \tag{10}$$

the coefficients W_k are given by

$$W_1=l^*/l,\qquad W_k=\frac{1}{\pi(k-1)}\sin\frac{\pi(k-1)l^*}{l},\quad k=\overline{2,M}.$$

As an example, let us arrange a heat-generating film between the first and second layers of a strip. Such arrangement of the film is usual for heated laminated glasses of vehicles.

Considering the system of conditions (7)–(9) as a system of linear algebraic equations in $a_{ik}(t)$, $b_{ik}(t)$ and $c_{ik}(t)$

$$[\boldsymbol{\Gamma}]\mathbf{V}=\mathbf{Q}, \tag{11}$$

for an evaluation of the right-hand sides of Eq. (5) in the course of integration of these equations in time, solve the system (11) on each process step.

Construct the matrix and vector of the right-hand side of the system (11)

$$\gamma_{11} = \alpha_t, \quad \gamma_{12} = \sqrt{3}(\alpha_t + 2\lambda_1/h_1), \quad \gamma_{13} = \sqrt{5}(\alpha_t + 6\lambda_1/h_1),$$

$$\gamma_{22} = -2\sqrt{3}\lambda_1/h_1, \quad \gamma_{23} = 6\sqrt{5}\lambda_1/h_1, \quad \gamma_{24} = 2\sqrt{3}\lambda_2/h_2, \quad \gamma_{25} = 6\sqrt{5}\lambda_2/h_2,$$

$$\gamma_{31} = 1, \quad \gamma_{32} = -\sqrt{3}, \quad \gamma_{33} = \sqrt{5}, \quad \gamma_{34} = -1, \quad \gamma_{35} = -\sqrt{3}, \quad \gamma_{36} = -\sqrt{5},$$

$$\gamma_{45} = -2\sqrt{3}\lambda_2/h_2, \quad \gamma_{46} = 6\sqrt{5}\lambda_2/h_2, \quad \gamma_{48} = 2\sqrt{3}\lambda_3/h_3, \quad \gamma_{49} = 6\sqrt{5}\lambda_3/h_3,$$

$$\gamma_{54} = 1, \quad \gamma_{52} = -\sqrt{3}, \quad \gamma_{53} = \sqrt{5}, \quad \gamma_{54} = -1, \quad \gamma_{55} = -\sqrt{3}, \quad \gamma_{56} = -\sqrt{5}, \ldots,$$

$$\gamma_{2N-2,3N-4} = -2\sqrt{3}\lambda_{N-1}/h_{N-1}, \quad \gamma_{2N-2,3N-3} = 6\sqrt{5}\lambda_{N-1}/h_{N-1},$$

$$\gamma_{2N-2,3N-1} = 2\sqrt{3}\lambda_N/h_N, \quad \gamma_{2N-2,N} = 6\sqrt{5}\lambda_N/h_N, \quad \gamma_{2N-1,3N-5} = 1,$$

$$\gamma_{2N-1,3N-4} = -\sqrt{3}, \quad \gamma_{2N-1,3N-3} = \sqrt{5}, \quad \gamma_{2N-1,3N-2} = -1,$$

$$\gamma_{2N-1,3N-1} = -\sqrt{3}, \quad \gamma_{2N-1,3N} = -\sqrt{5}, \quad \gamma_{2N,3N-2} = \alpha_b,$$

$$\gamma_{2N,3N-1} = -\sqrt{3}(\alpha_b + 2\lambda_N/h_N), \quad \gamma_{2N,3N} = \sqrt{5}(\alpha_b + 6\lambda_N/h_N),$$

$$q_1 = \alpha_t T_t, \quad q_2 = W_k, \quad q_{2N} = \alpha_b T_b.$$

The remaining components of matrix $[\boldsymbol{\Gamma}]$ and vector of the right-hand side $\mathbf{Q}$ are equal to zero. The components of vector $\mathbf{V}$ have the form

$$v_1 = a_{1k}, \quad v_2 = b_{1k}, \quad v_3 = c_{1k}, \quad \ldots, \quad v_{N-2} = a_{Nk}, \quad v_{N-1} = b_N, \quad v_N = c_{Nk}.$$

The matrix of system $[\boldsymbol{\Gamma}]$ contains $2N$ rows and $3N$ columns. We express parameters $b_{ik}(t)$ and $c_{ik}(t)$ through functions $a_{ik}(t)$ and vector $\mathbf{Q}$. Transposing the columns of the matrix being coefficients at $a_{ik}(t)$ (first, fourth, seventh and etc.) to the right-hand side of the system (11), we form a matrix $[\mathbf{B}]$ from them. Denote those retained on the left-hand side, as the matrix $[\mathbf{A}]$ of dimension $2N \times 2N$, and obtain a system

$$[\mathbf{A}]\mathbf{Y} = [\mathbf{B}]\mathbf{X} + \mathbf{Q},$$

where $\mathbf{Y}$ is vector with components $b_{ik}(t)$ and $c_{ik}(t)$ $\left(i = \overline{1, N}\right)$, and $\mathbf{X}$ is vector from coefficients $a_{ik}(t)$. The solution of this system can be written as

$$\mathbf{Y} = \lfloor \mathbf{A}^* \rfloor \mathbf{X} + \mathbf{Q}^*, \tag{12}$$

where

$$\lfloor \mathbf{A}^* \rfloor = [\mathbf{A}]^{-1}[\mathbf{B}], \qquad \mathbf{Q}^* = [\mathbf{A}]^{-1}\mathbf{Q}.$$

The matrix $\lfloor \mathbf{A}^* \rfloor$ has $2N$ rows and N columns; the vector $\mathbf{Q}^*$ contains $2N$ components.

Now let us return to Cauchy problem (5) and reduce it to the standard form. For this purpose, we obtain matrix $\lfloor \hat{\mathbf{A}} \rfloor$ and vector $\hat{\mathbf{Q}}$ from even rows of $\lfloor \mathbf{A}^* \rfloor$ and $\mathbf{Q}^*$ by multiplying each of rows by $v_i 12\sqrt{5}\lambda_2/h_i^2$, $i = \overline{1, N}$. Also, expression $v_i \alpha_k^2$ is subtracted from diagonal elements of matrix $\lfloor \hat{\mathbf{A}} \rfloor$. Then the system (5) takes the form

$$\dot{\mathbf{X}} = \lfloor \hat{\mathbf{A}} \rfloor \mathbf{X} + \hat{\mathbf{Q}}. \tag{13}$$

Functions $b_{ik}(t)$ and $c_{ik}(t)$ are obtained from relationship (12).

The system (13) is integrated by a modified method of expanding the solution into a Taylor series (Shupikov and Ugrimov, 1999). Note that if the components of matrix $\lfloor \hat{\mathbf{A}} \rfloor$ and vector $\hat{\mathbf{Q}}$ do not depend on time, by virtue of the properties of matrix $\lfloor \hat{\mathbf{A}} \rfloor$, the solution of the problem (13) approaches the solution of a steady-state problem. In this case the coefficients of expansion (2) for polynomials, higher than the first degree, tend to zero with time.

3. Non-stationary heat conduction in a laminated plate

Now consider a rectangular laminated plate assembled from N isotropic layers. The geometrical dimensions of a plate in direction of coordinate axes $0x$ and $0y$ are designated as l_x and l_y, respectively.

The heat conduction equation for ith layer of a plate can be written as

$$v_i \Delta T = \frac{\partial T}{\partial t}, \quad i = 1, 2, \ldots, n, \ \ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{14}$$

We consider the problem as symmetrical with regards to coordinate axes $0x$ and $0y$. Consequently, the solution of Eq. (14) can be represented in the form (2)

$$T_i(x, y, z_i, t) = \sum_{k=1}^{M} \sum_{l=1}^{M} \left[a_{kl}^i(t) f_1(\bar{z}) + b_{kl}^i(t) f_2(\bar{z}) + c_{kl}^i(t) f_3(\bar{z}) \right] \cos(\alpha_{xk} x) \cos(\alpha_{yl} y), \tag{15}$$

where

$$0 \leqslant x \leqslant l_x/2, \quad 0 \leqslant y \leqslant l_y/2, \quad \alpha_{xk} = 2\pi(k-1)/l_x, \quad \alpha_{yl} = 2\pi(l-1)/l_y.$$

Projection of Eq. (15) into functions f_r Eq. (3), in view of the function orthogonality condition (4), yields a system of differential equations

$$\dot{a}_{kl}^i = v_i \frac{12\sqrt{5}}{h_i^2} c_{kl}^i - v_i \left(\alpha_{xk}^2 + \alpha_{yl}^2 \right) a_{kl}^i, \quad \dot{b}_{kl}^i = -v_i \left(\alpha_{xk}^2 + \alpha_{yl}^2 \right) b_{kl}^i, \quad \dot{c}_{kl}^i = -v_i \left(\alpha_{xk}^2 + \alpha_{yl}^2 \right) c_{kl}^i. \tag{16}$$

Writing down the boundary conditions of convective heat transfer over top and bottom plate surfaces, we obtain

$$\frac{\lambda_1}{h_1} \left(b_{kl}^1 2\sqrt{3} + c_{kl}^1 6\sqrt{5} \right) = \alpha_t \left(T_t - a_{kl}^1 - b_{kl}^1 \sqrt{3} - c_{kl}^1 \sqrt{5} \right),$$
$$\frac{\lambda_N}{h_N} \left(b_{kl}^N 2\sqrt{3} - c_{kl}^N 6\sqrt{5} \right) = \alpha_b \left(T_b - a_{kl}^N + b_{kl}^N \sqrt{3} - c_{kl}^N \sqrt{5} \right),$$

as well as conditions of equality of heat flows and temperatures on interfaces of layers

$$-\frac{\lambda_i}{h_i} \left(b_{kl}^i 2\sqrt{3} - c_{kl}^i 6\sqrt{5} \right) + \frac{\lambda_{i+1}}{h_{i+1}} \left(b_{kl}^{i+1} 2\sqrt{3} + c_{kl}^{i+1} 6\sqrt{5} \right) = 0,$$
$$a_{kl}^i - b_{kl}^i \sqrt{3} + c_{kl}^i \sqrt{5} - a_{kl}^{i+1} - b_{kl}^{i+1} \sqrt{3} - c_{kl}^{i+1} \sqrt{5} = 0.$$

For each pair of harmonics, k and l, a system of linear algebraic equations in coefficients $a_{kl}^i(t)$, $b_{kl}^i(t)$ and $c_{kl}^i(t)$ similar to the system (11) is formed. The coefficient of expansion of a function of intensity for heat source $W(x, y)$ (heat-generating film occupying rectangular area: $0 \leqslant |x| \leqslant l_x^*/2$, $0 \leqslant |y| \leqslant l_y^*/2$) is transformed to the form

$$W_k = \frac{4}{l_x l_y} \int_0^{l_x^*/2} \int_0^{l_y^*/2} W(x,y) \cos(\alpha_{xk} x) \cos(\alpha_{yl} y)\, \mathrm{d}x\, \mathrm{d}y. \tag{17}$$

The comparison of basic relations for a laminated strip (5) and plate (16) displays that a distinction exits in the formation of matrix $\lfloor \hat{\mathbf{A}} \rfloor$; from its diagonal elements, instead of coefficients $v_i \alpha_k^2$, coefficients $v_i\left(\alpha_{xk}^2 + \alpha_{yl}^2\right)$ are subtracted. The obtained system of differential equations is solved numerically by method of expansion of the solution in a Taylor series similar to system (13).

4. Numerical examples

The functionality of the method offered is illustrated by considering the heating of five-layer strips and rectangular plates. The material properties of the layers are presented in Table 1. The following designations are introduced: *A* is silicate glass, *B* is polyvinylbutyral, and *C* is acrylic.

There is a convective heat transfer over top and bottom surfaces of strips and plates. Convective heat transfer coefficients and temperature of environments over the top and bottom surfaces of a package are as follows: $\alpha_t = 80$ W/m^2 K, $\alpha_b = 25$ W/m^2 K, $T_t = 257$ K and $T_b = 293$ K. The initial temperature of layers is taken to be 273 K. The geometrical parameters of a strip are: $l = 0.9$ m, $h_1 = 5 \times 10^{-3}$ m, $h_2 = 3 \times 10^{-3}$ m, $h_3 = 1.5 \times 10^{-2}$ m, $h_4 = 2 \times 10^{-3}$ m, $h_5 = 2 \times 10^{-2}$ m.

For confirmation of reliability of the results obtained by the proposed method, let us compare them to the calculation performed by the finite element method (FEM). A strip with the composition *A–B–A–B–A* is considered. The source power is zero.

Fig. 2 shows the temperature distributions obtained by the proposed method and FEM, for some instances. The results are in good agreement, proving the accuracy of the method.

In the case where a strip contains a heat-generating film, the results of calculations are also very closely spaced. The maximum error does not exceed 0.5%. However, the simulation of heat generation in the film by FEM requires application of additional computational (implicit) techniques, as in the finite element formulation, the heat-generating layer should have a finite thickness. In the method proposed by the authors, the presence of a heat-generating layer is taken into account precisely in a condition of equality of heat flows at interfaces of layers (8).

Consider now the thermal condition of strips and plates containing a heat-generating film with power $W = 3500$ W/m^2, $l^*/l = 0.75$, $l_x^*/l_x = l_y^*/l_y = 0.75$ (see Eqs. (10) and (17)). As it has been specified above, we arrange the film between the first and the second layers of strips or plates.

Fig. 3 shows the temperature distribution through the thickness of the strip with the composition *A–B–A–B–A* at different instants of time in the central point $x = 0$. The dash–dot line designates the position of the heat-generating film in the package of layers. At $t = 1$ s the temperature distribution is non-linear through the thickness of layers, which is appreciable especially in the third and the fifth layers. With time, the temperature distribution in all layers also becomes linear (curve 3, $t = 10^3$ s). Quadratic terms in the set of functions (3) essentially influence on the problem solution (2) during the time interval $0 \leqslant t < 10^3$ s. A

Table 1
Material properties of layers

Property	Material		
	A	*B*	*C*
λ_i (W/m K)	1.60	0.17	0.21
c_i (kJ/kg K)	0.75	1.50	1.40
ρ_i (kg/m^3)	2500	1200	1200

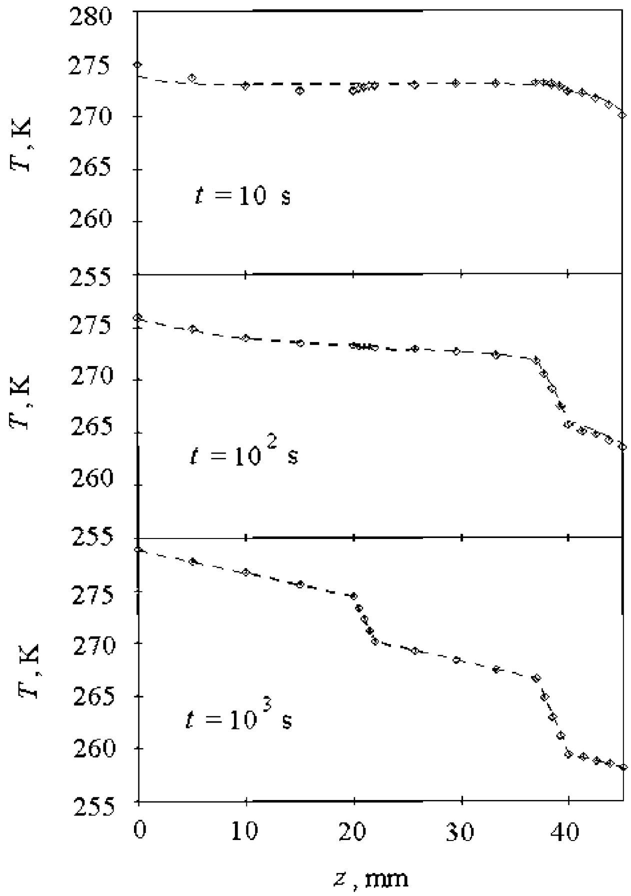

Fig. 2. Temperature distributions through the thickness of a strip at point $x = 0$ obtained by the proposed method (◇ ◇ ◇) and FEM (– – –); composition *A–B–A–B–A*.

large temperature gradient is observed in layers close to the surface with the heat-generating film ($n = 2$). Since $t = 5 \times 10^3$ s the temperature field does not practically vary, i.e. it stays steady. In the exact solution of a steady problem the temperature varies through the thickness of layers linearly. In this case the system of differential equations (5) is reduced to a system of linear algebraic equations.

Fig. 4 shows the similar distribution for the strip with the composition *A–B–C–B–C*. The distribution is non-linear through layer thicknesses, except for the first one. A large temperature gradient near the heated surface (curves 2 and 3) can result in significant temperature stresses within the strip. At $t = 10^4$ s the temperature field becomes a stationary one. Thus, material *C* hinders fast heat transfer.

Figs. 5 and 6 show the temperature variation along the length of strips on the surface with the film for compositions *A–B–A–B–A* and *A–B–C–B–C*, respectively. An increased temperature gradient along spatial coordinate x is observed near the edges of the film (the dash–dot line), that can result in significant temperature stresses in layers in this area. The temperature field is steady for compositions *A–B–A–B–A* and *A–B–C–B–C* since $t = 5 \times 10^3$ s and $t = 10^4$ s, respectively.

Temperature profiles over other surfaces of strips, as a whole, follow the obtained distributions but smooth out in the area of large temperature gradients along the spatial coordinate.

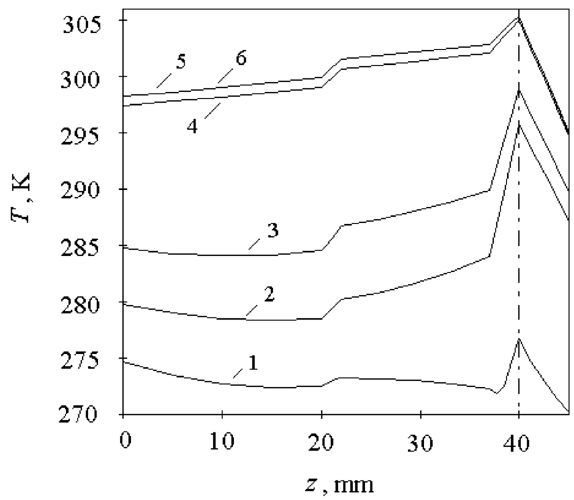

Fig. 3. Distribution of temperature T through the thickness of a strip at point $x = 0$ at different instants of time: $1 - t = 1$ s, $2 - t = 5 \times 10^2$ s, $3 - t = 10^3$ s, $4 - t = 5 \times 10^3$ s, $5 - t = 10^4$ s, $6 - t = 2 \times 10^4$ s; composition *A–B–A–B–A*.

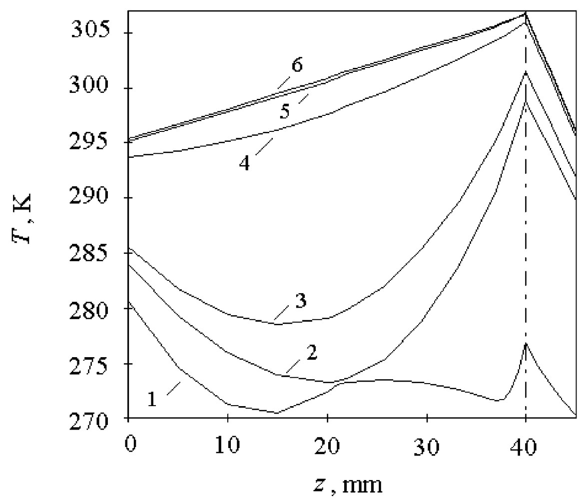

Fig. 4. Distribution of temperature T through the thickness of a strip at point $x = 0$ in different instants of time: $1 - t = 1$ s, $2 - t = 5 \times 10^2$ s, $3 - t = 10^3$ s, $4 - t = 5 \times 10^3$ s, $5 - t = 10^4$ s, $6 - t = 2 \times 10^4$ s; composition *A–B–C–B–C*.

In the case of plates, two compositions are considered as well. Thicknesses of layers coincide with the previous values taken for strips, and the lengths of sides are equal to 0.4 m. The analysis of calculations has shown that the temperature distributions through thickness of plates are similar to distributions for strips and, therefore, are not presented. Figs. 7 and 8 show the temperature distributions over the surface with the film at $t = 10^3$ s for compositions *A–B–A–B–A* and *A–B–C–B–C*, respectively.

A comparison of the results for the first and second compositions both for strips and for plates displays that the first composition allows a more uniform temperature distribution to be reached, and the temperature field reaches its steady state faster. In structures with composition *A–B–C–B–C* temperature

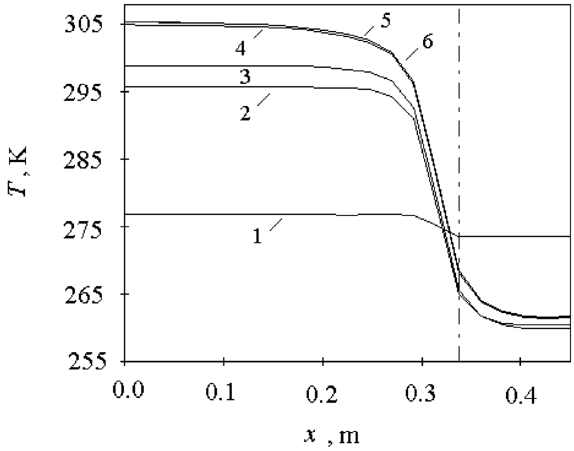

Fig. 5. Variation of temperature T along length of a strip ($x \in [0, l/2]$) at different instants of time: $1 - t = 1$ s, $2 - t = 5 \times 10^2$ s, $3 - t = 10^3$ s, $4 - t = 5 \times 10^3$ s, $5 - t = 10^4$ s, $6 - t = 2 \times 10^4$ s; composition *A–B–A–B–A*. surface $n = 2$.

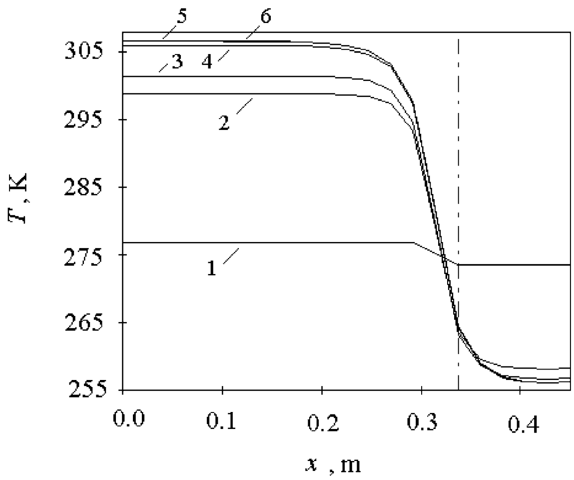

Fig. 6. Variation of temperature T along length of a strip ($x \in [0, l/2]$) at different instants of time: $1 - t = 1$ s, $2 - t = 5 \times 10^2$ s, $3 - t = 10^3$ s, $4 - t = 5 \times 10^3$ s, $5 - t = 10^4$ s, $6 - t = 2 \times 10^4$ s; composition *A–B–C–B–C*; surface $n = 2$.

gradients through thickness and length of strips or plates are greater than in structures with composition *A–B–A–B–A*. It results in high temperature stresses, which must be taken into account at evaluation of functionality of a real glazing.

5. Conclusions

A new method of solution of non-stationary heat conduction problem in laminated strips and plates is proposed. The transient temperature change is caused by an impulse action of a distributed heat source simulating a heat-generating film. Temperature distribution through the thickness of each layer is repre-

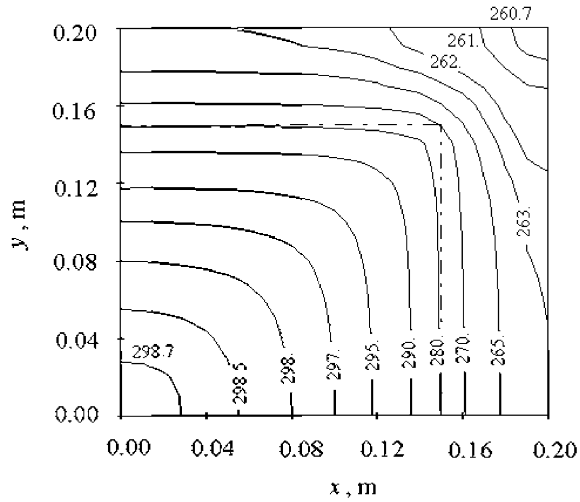

Fig. 7. Temperature distribution over the surface of a plate ($n = 2, x \in [0, l/2], y \in [0, l/2]$) at the instant of time $t = 10^3$ s; composition *A–B–A–B–A*.

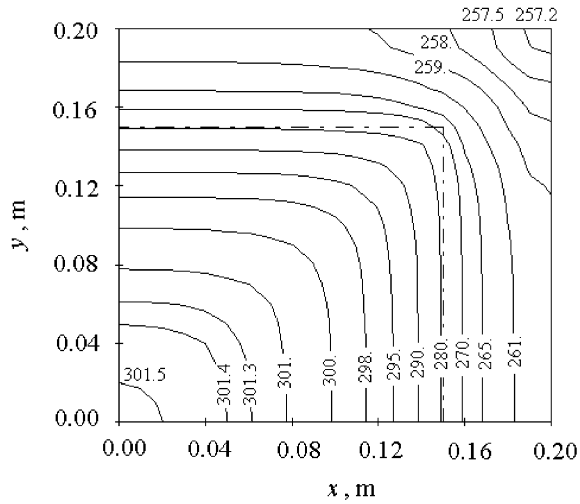

Fig. 8. Temperature distribution over the surface of a plate ($n = 2, x \in [0, l/2], y \in [0, l/2]$) at the instant of time $t = 10^3$ s; composition *A–B–C–B–C*.

sented by using Legendre orthonormal polynomials, which allows authentic (with a necessary accuracy) description of the thermal condition of laminated elements assembled from layers with different mechanical and geometrical characteristics.

The temperature fields are investigated using some examples of five-layer strips and plates. The change of heat-generating film power, and temperature of environment, can be given by any function of coordinates and time. The comparison of calculation results obtained by the proposed method to those obtained by FEM has confirmed their reliability.

The solution of such problems has practical importance, as the results of this research can be applied, for example, to the analysis of the efficiency of de-icing and de-misting performances of heating systems for windshields.

References

Heuer, R., Irschik, H., Zieler, F., 1992. Thermically forced vibrations of moderately thick polygonal plates. Journal of Thermal Stresses 15 (2), 203–210.

Jackson, D., 1948. Fourier Series and Orthogonal Polynomials. State Publishers of Foreign Literature, Moscow.

Kantor, B.Ya., Smetankina, N.V., Shupikov, A.N., 2000. Non-stationary heat conduction in layered medium. An one-dimensional case. Bulletin of the Kharkov State Polytechnic University, Technologies in Engineering 104, 114–118 (in Russian).

Khdeir, A.A., 1997. On the thermal response of antisymmetric angle-ply laminated plates. ASME Journal of Applied Mechanics 64 (1), 229–233.

Nusier, S.Q., Newaz, G.M., 1998. Transient residual stresses in thermal barrier coatings: analytical and numerical results. ASME Journal of Applied Mechanics 65 (2), 346–353.

Reddy, J.N., Hsu, Y.S., 1980. Effects of shear deformation and anisotropy on the thermal bending of layered composite plates. Journal of Thermal Stresses 3 (3), 475–493.

Reddy, J.N., Chin, C.D., 1998. Thermomechanical analysis of functionally graded cylinders and plates. Journal of Thermal Stresses 21 (6), 593–626.

Santhosh, U., 1992. Thermal shock response of layered orthotropic media. Journal of Thermal Stresses 15 (3), 339–353.

Savoia, M., Reddy, J.N., 1995. Three-dimensional thermal analysis of laminated composite plate. International Journal of Solids and Structures 32 (5), 539–608.

Shuji, I., Masataka, T., 1991. Analysis of two-dimensional transient thermoelasticity by the time-stepping boundary element method. Transactions of the JSME A 57 (544), 2999–3004.

Shupikov, A.N., Ugrimov, S.V., 1999. Vibrations of multilayer plates under the effect of impulse loads. Three-dimensional theory. International Journal of Solids and Structures 36 (22), 3391–3402.

Tanigawa, Y., Murakami, H., Ootao, Y., 1991a. Transient thermal stress analysis of a laminated composite beam. Journal of Thermal Stresses 12 (1), 25–39.

Tanigawa, Y., Ootao, Y., Kawamura, R., 1991b. Thermal bending of laminated composite rectangular plates and nonhomogeneous plates due to partial heating. Journal of Thermal Stresses 14 (3), 285–308.

Verijenko, V.E., Tauchert, T.R., Shaikh, C., Tabakov, P.Y., 1999. Refined theory of laminated anisotropic shells for the solution of thermal stress problems. Journal of Thermal Stresses 22 (1), 75–100.

Wu, C.H., Tauchert, T.R., 1980. Thermoelastic analysis of laminated plates. 2: Antisymmetric cross-ply and angle-ply laminates. Journal of Thermal Stresses 3 (3), 365–378.